

SYSTOLIC VOLUME AND COMPLEXITY OF 3-MANIFOLDS

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ABSTRACT. Let M be an orientable closed irreducible 3-manifold. We prove that if M is aspherical, the systolic volume of M , denoted $\text{SR}(M)$, is bounded below in terms of the complexity. This result shows that the systolic volume of 3-manifolds has the finiteness property. For any positive real number T , there are only a finite number of closed irreducible aspherical 3-manifolds M with $\text{SR}(M) < T$.

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1. INTRODUCTION

Let M be an n -dimensional manifold endowed with a Riemannian metric \mathcal{G} , denoted (M, \mathcal{G}) . Denote by $\text{Vol}_{\mathcal{G}}(M)$ the volume of (M, \mathcal{G}) . The homotopy 1-systole of (M, \mathcal{G}) , denoted $\text{Sys } \pi_1(M, \mathcal{G})$, is defined to be the length of the shortest noncontractible loop in M .

A topological space V is aspherical if all higher homotopy groups $\pi_i(V)$ vanish, with $i \geq 2$. The universal covering space of an aspherical manifold is contractible. The aspherical space V is an Eilenberg-MacLane $K(G, 1)$ space, where $G = \pi_1(V)$. A closed n -dimensional manifold M is essential if there exists a map $f : M \rightarrow K$ from M to

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an aspherical topological space K , such that $f_*([M])$ is nontrivial in $H_n(K(G, 1); R)$, where $[M] \in H_n(M; R)$ stands for the fundamental class. The coefficient ring R is \mathbb{Z} if M is orientable, and \mathbb{Z}_2 if M is nonorientable. Examples of essential manifolds include all closed aspherical manifolds, and real projective spaces in any dimension. Moreover, the connected sum $M \# M'$ is essential, if M is a closed essential n -dimensional manifold, and M' is any closed n -dimensional manifold.

Definition 1.1. Let M be a closed essential n -dimensional manifold. The systolic volume of M , denoted $\text{SR}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\text{Vol}_{\mathcal{G}}(M)}{\text{Sys } \pi_1(M, \mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics \mathcal{G} on M .

Babenko [2, Theorem 8.1.] showed that the systolic volume $\text{SR}(M)$ is a homotopy invariant of the essential manifold M .

Let M be a closed irreducible 3-manifold. The complexity of M , denoted $c(M)$, is the minimum number of tetrahedra in a pseudo-simplicial triangulation of the manifold. In this paper, we show that the systolic volume of an aspherical 3-manifold is bounded below in terms of the complexity.

Theorem 1.2. *Let M be an orientable closed irreducible 3-manifold. If M is aspherical, then we have*

$$\text{SR}(M) \geq C \frac{c(M)}{\exp\left(C' \sqrt{\log(5c(M))}\right)}, \quad (1.1)$$

where C and C' are two explicitly given positive constants.

If M is a nonorientable aspherical 3-manifold, we consider its orientable double cover \tilde{M} . It is easy to see that $\text{SR}(M) \geq \text{SR}(\tilde{M})$. Theorem 1.2 is applied to \tilde{M} to derive an estimate of $\text{SR}(M)$ similar to (1.1).

A closed orientable 3-manifold is aspherical if and only if it is irreducible and has infinite fundamental group. In particular, hyperbolic 3-manifolds are aspherical. The fundamental group determines a closed aspherical 3-manifold up to homeomorphism, cf. Lück [16, Remark 4.5].

The complexity invariant of 3-manifolds has the finiteness property, cf. Proposition 2.2 in Section 2. Hence Theorem 1.2 indicates that the systolic volume of aspherical 3-manifolds has the finiteness property. For any positive number T , there are only a finite number of aspherical 3-manifolds M with $\text{SR}(M) < T$.

Gromov [10, Theorem 0.1.A.] proved that if M is a closed essential n -dimensional manifold,

$$\text{Sys } \pi_1(M, \mathcal{G})^n \leq C_n \text{Vol}_{\mathcal{G}}(M), \quad (1.2)$$

where C_n is a constant only depending on n ,

$$0 < C_n < \left(6(n+1)n^n \cdot \sqrt{(n+1)!} \right)^n.$$

Conversely, Babenko [2, Corollary 8.3.] showed that if $\text{SR}(M) > 0$ for a closed orientable manifold M , then M is essential.

Gromov's systolic inequality (1.2) implies that $\text{SR}(M) > \frac{1}{C_n}$ for closed essential n -dimensional manifolds M . There are various investigations of the systolic volume $\text{SR}(M)$. A closed surface Σ is essential if it is not homeomorphic to S^2 . Gromov [10, Corollary 5.2.B.] proved that $\text{SR}(\Sigma) \geq \frac{3}{4}$ if Σ is a closed aspherical surface. Pu [20] showed that $\text{SR}(\mathbb{RP}^2) = \frac{2}{\pi}$. Hence we have

$$\text{SR}(\Sigma) \geq \frac{2}{\pi}$$

if Σ is a closed essential surface. Moreover, we know $\text{SR}(\mathbb{T}^2) = \frac{\sqrt{3}}{2}$ for the torus \mathbb{T}^2 , $\text{SR}(\mathbb{RP}^2 \# \mathbb{RP}^2) = \frac{2\sqrt{2}}{\pi}$ for the Klein bottle $\mathbb{RP}^2 \# \mathbb{RP}^2$. Currently we only know the exact values of systolic volumes of these three essential surfaces, see expositories in [8, 9, 14]. Nakamura [19, Theorem 5.1.] proved that if M is a closed essential 3-manifold other than the lens space $L(p, q)$ with odd order p , $\text{SR}(M) \geq \frac{1}{6}$.

Let Σ_g be a closed surface with genus $g \geq 2$, Gromov [11, 2.C.] proved that

$$\text{SR}(\Sigma_g) \geq C \frac{g}{(\log g)^2}, \quad (1.3)$$

where C is a universal positive constant independent of g . Katz and Sabourau [15, Theorem 4.1.] showed that asymptotically the constant C in (1.3) is π if genus g is sufficiently large. Let p be any odd prime number. Buser and Sarnak [7, Section 4.] showed that there exist arithmetic closed hyperbolic surfaces Σ_g with genus $g = p(p-1)(p+1)\nu + 1$, where $\nu > 0$ is a fixed constant, such that

$$\text{SR}(\Sigma_g) \leq \frac{9\pi}{4} \frac{g}{(\log g)^2}. \quad (1.4)$$

Therefore we conclude that when genus g of closed surfaces Σ_g is sufficiently large,

$$\pi \frac{g}{(\log g)^2} \lesssim \text{SR}(\Sigma_g) \lesssim C \frac{g}{(\log g)^2}, \quad (1.5)$$

where C is a constant satisfying $C \leq \frac{9\pi}{4}$. The asymptotic estimate (1.5) is generalized to essential manifolds M with dimension $n \geq 3$. Let $M_{\#k}$ denote the connected sum of k copies of M . Sabourau [21, Theorem A.] proved that

$$\text{SR}(M_{\#k}) \geq C_n \frac{k}{\exp(C'_n \sqrt{\log k})}, \quad (1.6)$$

where C_n and C'_n are two positive constants only depending on the dimension n . An upper bound of the systolic volume of connected sums is proved by Babenko and Balacheff [3],

$$\text{SR}(M_{\#k}) \leq C \frac{k \sqrt{\log \log k}}{\sqrt{\log k}} \quad (1.7)$$

if $k \geq N$ for some positive integer N , where C is a positive constant only depending on the manifold M .

Let M be a closed essential n -dimensional manifold, and let $c = \sum_i a_i \sigma_i$ be a cycle representing the fundamental class $[M] \in H_n(M; \mathbb{R})$. The simplicial volume of M , denoted $\|M\|_\Delta$, is defined to be

$$\inf_c |a_i|,$$

where the infimum runs over all cycles c representing $[M]$.

Theorem 1.3 (Gromov [10, Theorem 6.4.D']). *Let M be a closed essential n -dimensional manifold. Suppose that M has nonzero simplicial volume, then*

$$\text{SR}(M) \geq C_n \frac{\|M\|_\Delta}{\log^n(\|M\|_\Delta)}, \quad (1.8)$$

where C_n is a constant only depending on n .

A 3-manifold M has zero simplicial volume if and only if there are no hyperbolic pieces in its JSJ decomposition. However, the complexity $c(M)$ of a closed irreducible 3-manifold is always a positive number. Hence Theorem 1.2 covers more essential 3-manifolds. In terms of conjectures of Gromov [11, Section 3.C.], it is also reasonable to conjecture that

$$\text{SR}(M) \geq C \frac{c(M)}{\log^3(c(M))}$$

if M is a closed irreducible manifold other than S^3 , \mathbb{RP}^3 and $L(3, 1)$, where C is a constant.

We prove Theorem 1.2 in terms of the following estimation on orientable closed aspherical n -manifolds.

Proposition 1.4. *Let M be an orientable closed aspherical n -dimensional manifold. There exists a triangulation K of M , with t_n the number of n -simplices of K , such that*

$$\mathrm{SR}(M) \geq C_n \frac{t_n}{\exp(C'_n \sqrt{C''_n + \log t_n})} \quad (1.9)$$

where C_n, C'_n, C''_n are explicitly given constants only depending on n .

The inequality (1.9) is proved by applying the relevant estimation on systolic volume of homology classes, cf. Gromov [10, Section 6.], Babenko and Balacheff [4], Bulteau [5, 6].

The complexity of 3-manifolds has the additivity property on connected sums, that is, $c(M_1 \# M_2) = c(M_1) + c(M_2)$ for two closed irreducible 3-manifolds M_1 and M_2 . Hence Theorem 1.2 implies the following estimation.

Corollary 1.5. *Let M be an orientable, closed, irreducible, aspherical 3-manifold, and let $M_{\#k}$ be the connected sum of k copies of M . We have*

$$\mathrm{SR}(M_{\#k}) \geq \hat{C} \frac{k}{\exp(C' \cdot \sqrt{\log k})}, \quad (1.10)$$

where \hat{C} and C' are two positive constants which can be explicitly calculated.

Remark 1.6. The estimate (1.10) in Corollary 1.5 is implied by Theorem 1.2. This result is included in Sabourau [21, Theorem A]. In this paper it is verified in a different approach.

This paper is organized as follows. In Section 2, we introduce the complexity invariant of 3-manifolds. The complexity has the finiteness property and the additivity property on connected sums. Using Theorem 1.2, some known results of complexity are applied to derive lower bound estimations of the systolic volume. In Section 3, we introduce the systolic volume of homology classes. Gromov proved a lower bound estimation in terms of the number of simplices in a polyhedron representing the homology class. This lower bound estimation will be used in the proof of Proposition 1.4. In Section 4, the proof of Theorem 1.2 is given.

2. COMPLEXITY OF 3-MANIFOLDS

A triangulation of a 3-manifold M , denoted (\mathcal{T}, h) , is a simplicial complex \mathcal{T} with a homeomorphism $h : |\mathcal{T}| \rightarrow M$, where $|\mathcal{T}|$ stands for the union of all simplices of \mathcal{T} .

Definition 2.1 (Pseudo-simplicial triangulation, also see [13]). A pseudo-simplicial triangulation \mathcal{T} of a 3-manifold M contains

- (1) a set $\Delta = \{\tilde{\Delta}_i\}$ of disjoint collection of tetrahedra,
- (2) a family Φ of isomorphisms pairing faces of the tetrahedra in Δ so that if $\phi \in \Phi$, then ϕ is an orientation-reversing affine isomorphism from a face $\tilde{\sigma}_i \in \tilde{\Delta}_i$ to a face $\tilde{\sigma}_j \in \tilde{\Delta}_j$, possibly $i = j$.

Denote by Δ/Φ the space obtained from the disjoint union of the $\tilde{\Delta}_i$ by setting $x \in \tilde{\sigma}_i$ equal to $\phi(x) \in \tilde{\sigma}_j$, with the identification topology. The manifold M is homeomorphic to $|\mathcal{T}| = \Delta/\Phi$.

In a pseudo-simplicial triangulation, two faces of the same tetrahedra possibly are identified.

Let M be a closed irreducible 3-manifold. The complexity of M , denoted $c(M)$, is the minimum number of tetrahedra in a pseudo-simplicial triangulation of the manifold. This number agrees with the complexity defined by Matveev [17] if M is not homeomorphic to S^3 , \mathbb{RP}^3 and the lens space $L(3, 1)$.

Proposition 2.2 (Matveev [17, Theorem A, B]). (1) *For any integer k , there exist only a finite number of distinct closed irreducible orientable 3-manifolds of complexity k .*
 (2) *The complexity of the connected sum of compact 3-manifolds is equal to the sum of their complexities.*

The complexity of a 3-manifold measures how complicated a combinatorial description of the manifold must be, cf. [17]. There are various investigations of determining complexities of 3-manifolds, cf. [1, 18] and series of papers of Jaco, Rubinstein and Tillmann. Combined with Theorem 1.2, we can obtain lower bound estimations of systolic volumes of 3-manifolds. Some examples are listed as follows.

Example 2.3. (1) Let M_n be the total space of the \mathbb{T}^2 -bundle over S^1 with the monodromy A^n , where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Anisov [1] proved that $c(M_n) \geq 1.19n$ for all $n > 1$. The 3-manifold M_n has Sol geometry. So that Theorem 1.2 implies

$$\text{SR}(M_n) \geq C \frac{1.19n}{\exp(C' \sqrt{5.95n})},$$

where C and C' are explicitly given constants.

- (2) Let $n \geq 5$, and let $R(n)$ be the right-angled polytope in \mathbb{H}^3 with $(2n + 2)$ faces. Let $g_i \in \text{Isom}(\mathbb{H}^3)$ be the reflection in the plane containing the i -th face of $R(n)$. Denote by $G(n)$ the group

generated by $\{g_i\}_{i=1}^{2n+2}$. Löbell manifolds of order n , denoted $\mathcal{L}(n)$, are manifolds defined in the following way,

$$\mathcal{L}(n) = \left\{ \mathbb{H}^3 / \text{Ker}(\varphi) : \varphi : G(n) \rightarrow (\mathbb{Z}_2)^3 \text{ epimorphism,} \right. \\ \left. \text{Ker}(\varphi) < \text{Isom}^+(\mathbb{H}^3), \text{Ker}(\varphi) \text{ is torsion-free} \right\}.$$

Matveev et al. [18] proved that for sufficiently large n , the complexity of $M \in \mathcal{L}(n)$ satisfies $c(M) > 10n$. Hence Theorem 1.2 implies that

$$\text{SR}(M) \geq C \frac{10n}{\exp\left(C' \sqrt{\log(50n)}\right)},$$

where C and C' are explicitly given constants.

(3) Let $n \geq 4$. Denote by $F(2, n)$ the group

$$\langle x_1, x_2, \dots, x_n : x_i x_{i+1} x_{i+2}^{-1}, i = 1, \dots, n \rangle,$$

where indices are understood modulo n . The n -th Fibonacci manifold, denoted $M(n)$, is defined by $\mathbb{H}^3 / F(2, 2n)$. Matveev et al. [18] proved that for sufficiently large n , $c(M(n)) > 2n$. Hence Theorem 1.2 implies that

$$\text{SR}(M) \geq C \frac{2n}{\exp\left(C' \sqrt{\log(10n)}\right)},$$

where C and C' are explicitly given constants.

3. SYSTOLIC VOLUME OF HOMOLOGY CLASSES

A pseudomanifold of dimension n , cf. Spanier [22, Chapter 3], is an n -dimensional simplicial complex satisfying

- (1) Every simplex is a face of some n -simplex.
- (2) Every $(n-1)$ -simplex is the face of two n -simplices.
- (3) For every pair of n -simplices σ and σ' , there exist n -simplices $\sigma = \sigma_0, \sigma_1, \dots, \sigma_k = \sigma'$ such that $\sigma_i \cap \sigma_{i+1}$ is an $(n-1)$ -simplex for $0 \leq i < k$.

Let Γ be an arbitrary discrete group. There exists an aspherical space $K(\Gamma, 1)$ with $\pi_1(K(\Gamma, 1)) = \Gamma$. The n -th homology of group Γ , denoted $H_n(\Gamma)$, is defined to be $H_n(K(\Gamma, 1); R)$, where R stands for the coefficient ring \mathbb{Z} or \mathbb{Z}_2 . A geometric cycle representing a homology class $h \in H_n(K(\Gamma, 1); R)$, denoted (V, f, \mathcal{G}) , is an n -dimensional pseudomanifold V endowed with a piecewise smooth Riemannian metric \mathcal{G} , such that there exists a continuous map $f : V \rightarrow K(\Gamma, 1)$ with $f_*([V]) = h$.

The f -relative homotopy 1-systole of a geometric cycle (V, f, \mathcal{G}) , denoted $\text{Sys}_f \pi_1(V, \mathcal{G})$, is defined to be the length of the shortest loop γ in

V whose image $f(\gamma)$ is noncontractible in $K(\Gamma, 1)$. The systolic volume of a homology class $h \in H_n(\Gamma)$, denoted $\text{SR}(h)$, is defined to be

$$\inf_{(V, f, \mathcal{G})} \frac{\text{Vol}_{\mathcal{G}}(V)}{\text{Sys}_f \pi_1(V, \mathcal{G})^n},$$

where the infimum runs over all geometric cycles (V, f, \mathcal{G}) representing the homology class h . According to Gromov [10, Section 6], there exists a positive constant C_n only depending on the dimension n , such that

$$\text{SR}(h) \geq C_n.$$

Gromov [10, Section 6.4] defined regular geometric cycles in terms of the filling volume.

Lemma 3.1 (Gromov [10, Section 6.3, Section 6.4]). *Each homology class $h \in H_n(\Gamma)$ can be represented by a regular geometric cycle (V^*, f, \mathcal{G}^*) such that*

(1)

$$\frac{\text{Vol}_{\mathcal{G}^*}(V^*)}{\text{Sys}_f \pi_1(V^*, \mathcal{G}^*)} = \text{SR}(h).$$

(2) *For any $v \in V^*$, the ball $B(v, r) \subset V^*$ with the center v and the radius $r \leq \frac{1}{2} \text{Sys}_f \pi_1(V^*, \mathcal{G}^*)$ satisfies*

$$\text{Vol}_{\mathcal{G}^*}(B(v, r)) \geq A_n r^n, \quad (3.1)$$

where A_n is a constant only depending on n ,

$$A_n > n^{-n+2} \left((n-1)! \sqrt{n!} \right)^{-n+1}.$$

Let K be a topological space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{I}}$ be an open cover of K , where \mathcal{I} is the index set. A nerve of \mathcal{U} , denoted $\mathcal{N}_{\mathcal{U}}$, is a simplicial complex. For each open set U_α in \mathcal{U} , the nerve $\mathcal{N}_{\mathcal{U}}$ has a vertex v_α . If there is a nonempty intersection of $k+1$ open sets $\{U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_k}\}$ in the cover \mathcal{U} , the nerve $\mathcal{N}_{\mathcal{U}}$ has a k -simplex spanned by vertices $\{v_{\alpha_0}, v_{\alpha_1}, \dots, v_{\alpha_k}\}$.

Proposition 3.2 (Hatcher [12, 4G.3.]). *Let K be a paracompact space with an open cover \mathcal{U} . If every nonempty intersection of finitely many sets in \mathcal{U} is contractible, the nerve $\mathcal{N}_{\mathcal{U}}$ of \mathcal{U} is homotopy equivalent to K .*

Theorem 3.3 (cf. Gromov [10, Theorem 6.4.C']). *For each homology n -class $h \in H_n(K(\Gamma, 1); R)$, with $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$, there exist a polyhedron P with the number of k -simplices denoted s_k , and a map*

$g : P \rightarrow K(\Gamma, 1)$ sending the fundamental class $[P]$ to h through the induced homology homomorphism, and

$$\text{SR}(h) \geq C_n \frac{s_k}{\exp(C'_n \cdot \sqrt{C''_n + \log s_k})}, \quad (3.2)$$

where C_n, C'_n, C''_n are explicitly given constants only depending on n .

We show a detailed proof of Gromov's theorem in the following. Another complete proof can be found in Bulteau [5] or Bulteau [6]. Constants C_n, C'_n, C''_n in (3.2) can be explicitly calculated,

$$\begin{aligned} C_n &= 5^{\frac{(n-1)^2}{2}} \cdot \frac{1}{8^n} \cdot n^{-n+2} \left((n-1)! \sqrt{n!} \right)^{-n+1}, \\ C'_n &= (n+1) \sqrt{\log 5}, \\ C''_n &= \frac{(n-1)^2}{4} \log 5. \end{aligned}$$

Proof. Let Γ be a discrete group, and let (V^*, f, \mathcal{G}^*) be a regular geometric cycle representing the homology class $h \in H_n(K(\Gamma, 1); R)$, cf. Lemma 3.1. Let $R_0 = \frac{1}{8} \text{Sys}_f \pi_1(V^*, \mathcal{G}^*)$, and let $\alpha > 1$ be a constant. Assume that $0 < r < R_0$. A ball $B(x, r) \subset V^*$ with the center x and the radius r is called α -admissible, if

$$\text{Vol}_{\mathcal{G}^*}(B(x, 5r)) \leq \alpha \text{Vol}_{\mathcal{G}^*}(B(x, r)),$$

and

$$\alpha \text{Vol}_{\mathcal{G}^*}(B(x, r')) \leq \text{Vol}_{\mathcal{G}^*}(B(x, 5r'))$$

for any $r < r' \leq R_0$.

Let $k \geq 1$ be an integer. If we take α -admissible balls $B(v, r)$ with the center $x \in V$ and the radius r satisfying

$$5^{-k-1} R_0 \leq r \leq 5^{-k} R_0,$$

the volume of V^* can be estimated as follows,

$$\begin{aligned} \text{Vol}_{\mathcal{G}^*}(V^*) &\geq \text{Vol}_{\mathcal{G}^*}(B(v, R_0)) \\ &\geq \alpha \text{Vol}_{\mathcal{G}^*}(B(v, 5^{-1} R_0)) \\ &\vdots \\ &\geq \alpha^k \text{Vol}_{\mathcal{G}^*}(B(v, 5^{-k} R_0)) \\ &\geq \alpha^k \text{Vol}_{\mathcal{G}^*}(B(v, r)) \\ &\geq \alpha^k A_n r^n \\ &\geq \alpha^k A_n 5^{-nk-n} R_0^n. \end{aligned}$$

The inequality (3.1),

$$\text{Vol}_{\mathcal{G}^*}(B(v, r)) \geq A_n r^n,$$

is used in the above estimating process. Then we have

$$k \leq \frac{\log_5(\text{Vol}_{\mathcal{G}^*}(V^*)) - n \log_5 R_0 - \log_5 A_n + n}{\log_5 \alpha - n}. \quad (3.3)$$

We choose the constant α such that

$$\log_5 \alpha = n + \sqrt{\log_5(\text{Vol}_{\mathcal{G}^*}(V^*)) - n \log_5 R_0 - \log_5 A_n + n}. \quad (3.4)$$

On the pseudomanifold V^* we choose a maximal collection \mathcal{U}_1 of pairwise disjoint α -admissible open balls, denoted $\{B(v_i, r_i)\}_{i=1}^N$. The radius r_i of $B(v_i, r_i) \subset \mathcal{U}_1$ satisfies

$$5^{-k-1} R_0 \leq r_i \leq 5^{-k} R_0,$$

where k is the largest nonnegative integer satisfying the inequality (3.3). And the constant α is chosen by (3.4). Moreover, we assume $r_1 \geq r_2 \cdots \geq r_N$. The collection \mathcal{U}_1 is maximal in the sense that if \mathcal{U}' is another collection of α -admissible balls and $\mathcal{U}_1 \subset \mathcal{U}'$, then $\mathcal{U}_1 = \mathcal{U}'$. It is easy to verify the following facts,

- (1) The collection $\mathcal{U}_2 = \{B(v_i, 2r_i)\}_{i=1}^N$ is an open cover of V^* .
- (2) If $B(v_j, 2r_j) \cap B(v_{j_k}, 2r_{j_k}) \neq \emptyset$, with $j < j_k$ and $k = 1, 2, \dots, N_j$, then $B(v_{j_k}, r_{j_k}) \subset B(v_j, 5r_j)$ for all k .

Let P be the nerve of the open cover \mathcal{U}_2 . Since we have

$$2r_i < 2R_0 \leq \frac{1}{4} \text{Sys}_f \pi_1(V^*, \mathcal{G}^*),$$

each ball $B(v_i, 2r_i)$ is contractible. Hence the nerve P is homotopy equivalent to V^* , which is yielded by Proposition 3.2. There exists a map $g : P \rightarrow K(\Gamma, 1)$, such that $g_*([P]) = f_*([V^*]) = h$.

We have

$$\begin{aligned} \text{Vol}_{\mathcal{G}^*}(B(v_j, 5r_j)) &\geq \sum_{k=1}^{N_j} \text{Vol}_{\mathcal{G}^*}(B(v_{j_k}, r_{j_k})) \\ &\geq \sum_{k=1}^{N_j} A_n 5^{-nk-n} R_0^n \\ &= A_n 5^{-nk-n} R_0^n N_j. \end{aligned}$$

Then

$$\begin{aligned}
\text{Vol}_{\mathcal{G}^*}(V^*) &\geq \sum_{j=1}^N \text{Vol}_{\mathcal{G}^*}(B(v_j, r_j)) \\
&\geq \sum_{j=1}^N \alpha^{-1} \text{Vol}_{\mathcal{G}^*}(B(v_j, 5r_j)) \\
&\geq \alpha^{-1} \hat{N} A_n 5^{-nk-n} R_0^n,
\end{aligned}$$

where $\hat{N} = \sum_{j=1}^N N_j$, which is great than or equal to the number of k -simplices in the nerve P , with $k = 0, 1, \dots, n$. Hence we have

$$\begin{aligned}
\log_5 \hat{N} &\leq \log_5 (\text{Vol}_{\mathcal{G}^*}(V^*)) - n \log_5 R_0 - \log_5 A_n + n + kn + \log_5 \alpha \\
&\leq \log_5 (\text{Vol}_{\mathcal{G}^*}(V^*)) - n \log_5 R_0 - \log_5 A_n + n \\
&\quad + (n+1) \sqrt{\log_5 (\text{Vol}_{\mathcal{G}^*}(V^*)) - n \log_5 R_0 - \log_5 A_n + n + n},
\end{aligned}$$

so that

$$\log_5 \hat{N} \leq \log_5 u + (n+1) \sqrt{\log_5 u} + n, \quad (3.5)$$

where $u = \frac{40^n \text{Vol}(V^*)}{A_n \text{Sys } \pi_1(V^*, \mathcal{G}^*)^n}$. The inequality (3.5) yields that

$$\hat{N} \leq 5^n \cdot u \cdot 5^{(n+1) \sqrt{\log_5 u}}.$$

Therefore, we obtain the following estimate to the systolic ratio,

$$\frac{\text{Vol}_{\mathcal{G}^*}(V^*)}{\text{Sys } \pi_1(V^*, \mathcal{G}^*)^n} \geq \frac{5^{\frac{(n-1)^2}{2}} A_n}{8^n} \frac{\hat{N}}{\exp \left((n+1) \sqrt{\log 5} \sqrt{\frac{(n-1)^2}{4} (\log 5) + \log \hat{N}} \right)}. \quad (3.6)$$

Denote by s_k the number of k -simplices in the nerve P , with $k = 0, 1, \dots, n$. As mentioned above, we have $\hat{N} \geq s_k$. Hence the estimate (3.6) implies the inequality (3.2). \square

4. PROOF OF THE MAIN THEOREM

4.1. Proof of Proposition 1.4. Let M be a closed oriented aspherical n -dimensional manifold endowed with a Riemannian metric \mathcal{G} . Let $(V, f, \hat{\mathcal{G}})$ be a geometric cycle representing the fundamental class of M . In terms of the definition of geometric cycles, cf. Section 3., we have $f_*([V]) = [M]$, where $f_* : H_n(V; \mathbb{Z}) \rightarrow H_n(M; \mathbb{Z})$ is the induced homomorphism on homology groups, $[V] \in H_n(V; \mathbb{Z})$ and $[M] \in H_n(M; \mathbb{Z})$ are fundamental classes. Hence $f : V \rightarrow M$ is a degree one map.

Lemma 4.1. *There exists a pseudo-simplicial triangulation K of M , such that the number of n -simplices of K is less than or equal to the number of n -simplices of V .*

Proof. Let $\{\Delta_i\}_{i=1}^N$ be the set of all n -simplices of the pseudomanifold V . In simplicial homology, let $\sigma_i \in C_n(V)$ be the n -chain represented by Δ_i . The fundamental class $[V]$ is necessarily represented by the n -cycle $\sum_{i=1}^N a_i \sigma_i \in C_n(V)$, with $a_i = 1$ or -1 in terms of the orientation, cf. Hatcher [12, page 238.]. The homology homomorphism f_* is induced from the chain map $f_\# : C_n(V) \rightarrow C_n(M)$, and we have

$$f_\# \left(\sum_{i=1}^N a_i \sigma_i \right) = \sum_{i=1}^N a_i f_\#(\sigma_i).$$

Hence the n -chain $\sum_{i=1}^N a_i f_\#(\sigma_i)$ is a cycle in $C_n(M)$, representing the fundamental class $[M] \in H_n(M; \mathbb{Z})$. If for some i, j , $a_i = -a_j$ and $f_\#(\sigma_i) = f_\#(\sigma_j)$, then the actual number of n -chains in the cycle $\sum_{i=1}^N a_i f_\#(\sigma_i)$ is less than N . We use $\sum_{i=1}^{N'} a'_i \sigma'_i$ to denote the image cycle $\sum_{i=1}^N a_i f_\#(\sigma_i)$, which represents the fundamental class $[M] \in H_n(M; \mathbb{Z})$, and $N' \leq N$. In simplicial homology, the n -cycle $\sum_{i=1}^{N'} \sigma'_i$ is the combination of chains represented by all n -simplices in a pseudo-simplicial triangulation K with N' number of n -simplices. \square

According to Gromov [10, Section 6.3, Section 6.4], there exists a regular geometric cycle (V^*, f, \mathcal{G}^*) representing the fundamental class $[M] \in H_n(M; \mathbb{Z})$. The following properties are satisfied on (V^*, f, \mathcal{G}^*) ,

- (1) $\text{Vol}_{\mathcal{G}^*}(V^*) \leq \text{Vol}_{\mathcal{G}}(M)$,
- (2) $\text{Sys}_f \pi_1(V^*, \mathcal{G}^*) = \text{Sys} \pi_1(M, \mathcal{G})$.

Hence we have

$$\frac{\text{Vol}(M)}{\text{Sys} \pi_1(M, \mathcal{G})^n} \geq \frac{\text{Vol}(V^*)}{\text{Sys}_f \pi_1(V^*, \mathcal{G}^*)^n}.$$

Theorem 3.3 implies that

$$\frac{\text{Vol}(V^*)}{\text{Sys}_f \pi_1(V^*, \mathcal{G}^*)^n} \geq C_n \frac{s_n}{\exp(C'_n \sqrt{C''_n + \log s_n})},$$

where s_n is the number of n -simplices in the nerve $\mathcal{N}_{\mathcal{U}}$ of an open cover \mathcal{U} over V^* , and C_n, C'_n, C''_n are constants given in the inequality (3.2). Moreover, the nerve $\mathcal{N}_{\mathcal{U}}$ constructed is homotopy equivalent to V^* . There exists a map $\hat{f} : \mathcal{N}_{\mathcal{U}} \rightarrow M$ with $\hat{f}_*([\mathcal{N}_{\mathcal{U}}]) = [M]$. Lemma 4.1 yields that the number s_n of n -simplices of $\mathcal{N}_{\mathcal{U}}$ is greater than or equal to the number t_n of n -simplices in a pseudo-simplicial triangulation K of M . Hence the inequality (1.9) holds.

4.2. Proof of Theorem 1.2. We apply the estimation in Proposition 1.4 to closed aspherical 3-manifolds. Let M be an oriented, closed, irreducible and aspherical 3-manifold. Proposition 1.4 implies that there exists a triangulation K of M with t_3 the number of tetrahedra in K , such that

$$\mathrm{SR}(M) \geq C \frac{t_3}{\exp \left(C' \sqrt{\log (5t_3)} \right)}, \quad (4.1)$$

where C and C' are positive constants, $C > 0.0006$, $C' = 4\sqrt{\log 5}$. The complexity $c(M)$ is equal to the minimum number of tetrahedra in a pseudo-simplicial triangulation. Hence we have $t_3 \geq c(M)$. Then the inequality (1.1) of Theorem 1.2 is yielded by (4.1).

Remark 4.2. The proof of Corollary 1.5 is a direct application of Theorem 1.2. The complexity of 3-manifolds has additivity property, cf. Proposition 2.2. We have $c(M_{\#k}) = k \cdot c(M)$, so that

$$\begin{aligned} \mathrm{SR}(M_{\#k}) &\geq C \cdot \frac{k \cdot c(M)}{\exp \left(C' \sqrt{\log (5k \cdot c(M))} \right)} \\ &\geq C \cdot \frac{k \cdot c(M)}{\exp \left(C' \sqrt{\log (5c(M))} \right) \exp (C' \sqrt{\log k})}. \end{aligned}$$

Denote by \hat{C} the constant

$$\frac{C \cdot c(M)}{\exp \left(C' \sqrt{\log (5c(M))} \right)},$$

then the inequality (1.10) is derived.

REFERENCES

- [1] S. Anisov, Exact values of complexity for an infinite number of 3-manifolds. *Moscow Math. J* **5** (2005), no. 2, 305–310.
- [2] I. Babenko, Asymptotic invariants of smooth manifolds. *Izv. Ross. Akad. Nauk Ser. Mat.* **56** (1992), no. 4, 707–751.
- [3] I. Babenko and F. Balacheff, Géométrie systolique des sommes connexes et des revêtements cycliques. *Math. Ann.* **333** (2005), 157–180.
- [4] I. Babenko and F. Balacheff, Systolic volume of homology classes. *Algebr. Geom. Topol.* **15** (2015), no. 2, 733–767.
- [5] G. Bulteau, Systolic geometry and regularization technique. arXiv:1506.07848 [math.GT] (2015).
- [6] G. Bulteau, Regular geometric cycles. arXiv:1506.09051 [math.GT] (2015).
- [7] P. Buser and P. Sarnak, On the period matrix of a Riemann surface of large genus (with an Appendix by JH Conway and NJA Sloane). *Invent. Math.* **117** (1994), no. 1, 27–56.

- [8] L. Chen and W. Li, Systoles of surfaces and 3-manifolds. *Geometry and Topology of Submanifolds and Currents* **646** (2015), 61–80, Contemporary Mathematics, Amer. Math. Soc., Providence, RI.
- [9] C. Croke and M. Katz, Universal volume bounds in Riemannian manifolds. *Surveys in Differential Geometry*, vol. 8, Lectures on Geometry and Topology held in honor of Calabi, Lawson, Siu, and Uhlenbeck at Harvard University, May 3–5, 2002, edited by S. T. Yau, International Press, Somerville, MA, 2003, 109–137.
- [10] M. Gromov, Filling Riemannian manifolds. *J. Differential Geom.* **18** (1983), no. 1, 1–147.
- [11] M. Gromov, Systoles and intersystolic inequalities. *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, 1, 291–362.
- [12] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [13] W. Jaco and J. Rubinstein, 0-efficient triangulations of 3-manifolds. *J. Differential Geom.* **65** (2003), no. 1, 61–168.
- [14] M. Katz, *Systolic geometry and topology*. With an appendix by Jake P. Solomon. *Mathematical Surveys and Monographs*, 137. American Mathematical Society, Providence, RI, 2007.
- [15] M. Katz and S. Sabourau, Entropy of systolically extremal surfaces and asymptotic bounds. *Ergodic Theory Dynam. Systems* **25** (2005), no. 4, 1209–1220.
- [16] W. Lück, Survey on aspherical manifolds. *Proceedings of the 5-th European Congress of Mathematics Amsterdam, 14 -18 July, 2008*, editors: Andre Ran, Herman te Riele, Hermann and Jan Wiegerinck, EMS (2010), 53–82.
- [17] S. Matveev, Complexity theory of three-dimensional manifolds. *Acta Appl. Math.* **19** (1990), no. 2, 101–130.
- [18] S. Matveev, C. Petronio and A. Vesnin, Two-sided asymptotic bounds for the complexity of some closed hyperbolic three-manifolds. *Journal of the Australian Mathematical Society* **86** (2009), no. 2, 205–219.
- [19] K. Nakamura, On isosystolic inequalities for \mathbb{T}^n , \mathbb{RP}^n , and M^3 . *arXiv:1306.1617 [math.DG]* (2013).
- [20] P.M. Pu, Some inequalities in certain nonorientable Riemannian manifolds. *Pacific J. Math.* **2** (1952), 55–71.
- [21] S. Sabourau, Systolic volume of hyperbolic manifolds and connected sums of manifolds. *Geom. Dedicata* **127** (2007), 7–18.
- [22] E. Spanier, *Algebraic topology*. Springer-Verlag, New-York (1966).

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